Classification and Recurrence/Transience

Thomas Mountford

EPFL

March 8, 2021

Stopping Times

Definition: For a Markov Chain $\{X_n\}_{n\geq 0}$ on I, a stopping time $T:\Omega \to \mathbb{N}\cup\{\infty\}$ is a *Stoppping time* if for every n

$$\{T=n\} \;\in\; \sigma(X_0, X_1\cdots X_n) \;\equiv\; \mathcal{F}_n$$

(which is equivalent to $\forall n\{T=n\} \;\in\; \mathcal{F}_n$).
So for a stopping time T, for every n, there is a subset of I^{n+1}, $A(T,n)$ so that

$$\{T=n\} \;\equiv\; \{(X_0, X_1, \cdots X_n) \in\; A(T,n)\}.$$

Note that T is a random variable (in the extended sense).
Examples:
For $A \subset I$, $T = \inf\{n : X_n \in A\}$.
$T \;=\; \inf\{n : \exists 0 \leq i < j \leq n : X_i = X_j\}$.
T cannot be a stopping time if for $m > n$, there are two distinct vectors $\overline{x} = (x_0, x_1, \cdots x_m)$ and $\overline{y} = (y_0, y_1, \cdots y_m)$ so that $x_k = y_k \forall k \leq n$ and so that $\{X_i = x_i \forall i \leq m\} \implies \{T=n\}$ but $\{X_i = y_i \forall i \leq m\} \implies \{T \neq n\}$

Strong Markov Property

Theorem

For a stopping time T and $i \in I$, conditional on $T < \infty, X_T = i$, the process

$$(Z_n)_{n \geq 0} \equiv (X_{T+n})_{n \geq 0}$$

is a (δ_i, P) Markov chain conditionally independent of $(X_0, X_1 \cdots X_T)$

Recurrence

Definition: For a transition matrix on I, we say that site i is *recurrant* if

$$\mathbb{P}_i(T_i < \infty) = 1$$

where $T_i = \inf\{n \geq 1 : X_n = i\}$. If this is not so then i is said to be *transient*.

Lemma

i is recurrent if and only if

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 1$$

We introduce notation $T_i^0 = 0, \ T_i^{r+1} = \inf\{n > T_i^r : X_n = i\}$. Every T_i^r is a stopping time.

Proof.

If i is recurrent by the Strong Markov applied at stopping time $T = T_i^r$

$$\mathbb{P}_i(T_i^{r+1} < \infty \mid T_i^r < \infty) = \mathbb{P}_i(T_i < \infty) = 1$$

so by induction on r, $\mathbb{P}_i(T_i^r < \infty) = 1 \forall r$. □

Transience

Lemma

For i transient, then $\forall j \in I$

$$\mathbb{P}_j(T_i^r < \infty) \leq (\mathbb{P}_i(T_i < \infty))^{r-1}$$

Proof.

We use induction on r. The bound obviously holds for $r = 1$. Suppose it holds for r, then

$$\mathbb{P}_j(T_i^{r+1} < \infty \mid T_i^r < \infty) = \mathbb{P}_i(T_i < \infty)$$

and so
$\mathbb{P}_j(T_i^{r+1} < \infty) = \mathbb{P}_j(T_i^r < \infty)\mathbb{P}_i(T_i < \infty) = (\mathbb{P}_i(T_i < \infty))^r$. □

A necessary and sufficient condition

Theorem

A site $i \in I$ is recurrent if and only if

$$\sum_n p_{ii}^n = \infty$$

Proof.

$\sum_{r=1}^n p_{ii} = \mathbb{E}_i(\sum_{r=1}^n I_{X_r=i})$ is the expectation of the numbber of visits to$i$ up to and including time n. If i is recurrent, then as n tends to infinity, this expectation must tend to infinity. If i is transient, then the expectation(starting from i) of N_i, the number of visits (for $n \geq 1$) is exactly

$$\sum_r \mathbb{P}_i(T_i^r < \infty) = \sum_r \mathbb{P}_i(N \geq r) = \sum_r \mathbb{P}_i(T_i < \infty)^r$$

$= \frac{\mathbb{P}_i(T_i<\infty)}{1-\mathbb{P}_i(T_i<\infty)} < \infty$. And we have $\sum_{r=1}^\infty p_{ii} < \infty$.

Recurrent communicating classes

Consider a transition matrix P. This matrix partitions I into communicating classes and determines whether a site is recurrent or transient. In fact

Theorem

For a communicating class for P, either all sites are recurrent or all are transient.

Proof.

It is enough to show that if for communicating class C a site in it is recurrent then any other site $j \in C$ must be recurrent. Suppose for $i \in C$ that i is recurrent, so $\sum_n p_{ii}^n = \infty$. Given $j \in C$, by the definition of communicating class, there exist positive integers r and s so that $p_{ij}^r p_{ji}^s > 0$. Then

$$\sum_{n=1}^{\infty} p_{jj}^n \geq \sum_{n=r+s}^{\infty} p_{jj}^n \geq \sum_{k=1} p_{ji}^s p_{ii}^k p_{ij}^r$$

$= p_{ij}^r p_{ji}^s \sum_{k=1} p_{ii}^k \;=\; \infty.$

Given the preceding definition the following definitions make sense

a. A communicating class is said to be *transient* if a single site in it is transient. Equivalently if all sites in it are transient.

b. A communicating class is said to be *recurrent* if a single site in it is recurrent. Equivalently if all sites in it are recurrent.

c. A Markov chain is said to be recurrent if all sites are recurrent, it is said to be transient if all sites are transient. A Markov chain need not be recurrent or transient.

d. An irreducible Markov chain is *recurrent* if a site is recurrent (or equivalently if all sites are recurrent). Otherwise it is transient.